Lecture 11. Indefinite integral. Techniques of integration. Substitution in a definite integral. Integration by parts.

Integration Techniques

Most integrals involving "simple" substitutions will not have any of the substitution work shown. It is going to be assumed that you can verify the substitution portion of the integration yourself.

Here is a list of topics that are covered in this.

Integration by Parts – Of all the integration techniques covered in this chapter this is probably the one that students are most likely to run into down the road in other classes.

Integrals Involving Trig Functions – In this section we look at integrating certain products and quotients of trig functions.

Trig Substitutions – Here we will look using substitutions involving trig functions and how they can be used to simplify certain integrals.

Partial Fractions – We will use partial fractions to allow us to do integrals involving some rational functions.

Integrals Involving Roots – We will take a look at a substitution that can, on occasion, be used with integrals involving roots.

Integrals Involving Quadratics – In this section we are going to look at some integrals that involve quadratics.

Integration Strategy – We give a general set of guidelines for determining how to evaluate an integral.

Improper Integrals – We will look at integrals with infinite intervals of integration and integrals with discontinuous integrands in this section.

Comparison Test for Improper Integrals – Here we will use the Comparison Test to determine if improper integrals converge or diverge.

Approximating Definite Integrals – There are many ways to approximate the value of a definite integral. We will look at three of them in this section.

Integration by Parts

Let's start off with this section with a couple of integrals that we should already be able to do to get us started. First let's take a look at the following.

$$\int \mathbf{e}^{x}\,dx = \mathbf{e}^{x} + c$$

So, that was simple enough. Now, let's take a look at,

$$\int x\mathbf{e}^{x^2}\,dx$$

To do this integral we'll use the following substitution.

$$u = x^2 \qquad du = 2x\,dx \qquad \Rightarrow$$

$$\int x\mathbf{e}^{x^2}\,dx = \frac{1}{2}\int \mathbf{e}^{u}\,du = \frac{1}{2}\mathbf{e}^{u} + c = \frac{1}{2}\mathbf{e}^{x^2} +$$

Again, simple enough to do provided you remember how to do **substitutions**. By the way make sure that you can do these kinds of substitutions quickly and easily. From this point on we are going to be doing these kinds of substitutions in our head. If you have to stop and write these out with every problem you will find that it will take you significantly longer to do these problems.

Now, let's look at the integral that we really want to do.

$$\int x\mathbf{e}^{6x}\,dx$$

If we just had an *x* by itself or $\mathbf{e}^{6x}$ $\mathbf{e}^{6x}$ by itself we could do the integral easily enough. But, we don't have them by themselves, they are instead multiplied together.

There is no substitution that we can use on this integral that will allow us to do the integral. So, at this point we don't have the knowledge to do this integral.

To do this integral we will need to use integration by parts so let's derive the integration by parts formula. We'll start with the product rule.

$$\left(f\,g\right)' = f'\,g + f\,g'$$

Now, integrate both sides of this.

$$\int \left(f\,g\right)'\,dx = \int f'\,g + f\,g'\,dx$$

The left side is easy enough to integrate and we'll split up the right side of the integral.

$$fg = \int f'\,g\,dx + \int f\,g'\,dx$$

Note that technically we should have had a constant of integration show up on the left side after doing the integration. We can drop it at this point since other constants of integration will be showing up down the road and they would just end up absorbing this one.

Finally, rewrite the formula as follows and we arrive at the integration by parts formula.

$$\int f\,g'\,dx = fg - \int f'\,g\,dx$$

This is not the easiest formula to use however. So, let's do a couple of substitutions.

$$u = f\left(x\right) \qquad\qquad v = g\left(x\right)$$
$$du = f'\left(x\right)dx \qquad\qquad dv = g'\left(x\right)dx$$

Both of these are just the standard Calc I substitutions that hopefully you are used to by now. Don't get excited by the fact that we are using two substitutions here. They will work the same way.

Using these substitutions gives us the formula that most people think of as the integration by parts formula.

$$\int u\,dv = uv - \int v\,du$$

To use this formula we will need to identify *u* and *dv*, compute *du* and *v* and then use the formula. Note as well that computing *v* is very easy. All we need to do is integrate *dv*.

$$v = \int dv$$

So, let's take a look at the integral above that we mentioned we wanted to do.

Next, let's take a look at integration by parts for definite integrals. The integration by parts formula for definite integrals is,

Integration by Parts, Definite Integrals

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$

Note that the $uv\Big|_a^b$ in the first term is just the standard integral evaluation notation that you should be familiar with at this point. All we do is evaluate the term, *uv* in this case, at *b* then subtract off the evaluation of the term at *a*.

At some level we don't really need a formula here because we know that when doing definite integrals all we need to do is do the indefinite integral and then do the evaluation.

Let's take a quick look at a definite integral using integration by parts.

Since we need to be able to do the indefinite integral in order to do the definite integral and doing the definite integral amounts to nothing more than evaluating the indefinite integral at a couple of points we will concentrate on doing indefinite integrals in the rest of this section. In fact, throughout most of this chapter this will be the case. We will be doing far more indefinite integrals than definite integrals.

We will sometimes need more than one application of integration by parts to completely evaluate an integral. This is something that will happen so don't get excited about it when it does.

Actually, we didn't do anything wrong. We need to **remember** the following fact from Calculus I.

$$\text{If } f'(x) = g'(x) \text{ then } f(x) = g(x) +$$

In other words, if two functions have the same derivative then they will differ by no more than a constant. So, how does this apply to the above problem? First define the following,

$$f'(x) = g'(x) = x\sqrt{x+1}$$

Then we can compute $f(x)$ and $g(x)$ by integrating as follows,

$$f(x) = \int f'(x)\,dx \qquad\qquad g(x) = \int$$

We'll use integration by parts for the first integral and the substitution for the second integral. Then according to the fact $f(x)$ and $g(x)$ should differ by no more than a constant. Let's verify this and see if this is the case. We can verify that they differ by no more than a constant if we take a look at the difference of the two and do a little algebraic manipulation and simplification.

So, in this case it turns out the two functions are exactly the same function since the difference is zero. Note that this won't always happen. Sometimes the difference will yield a nonzero constant.
So just what have we learned? First, there will, on occasion, be more than one method for evaluating an integral. Secondly, we saw that different methods will often lead to different answers. Last, even though the answers are different it can be shown, sometimes with a lot of work, that they differ by no more than a constant.

When we are faced with an integral the first thing that we'll need to decide is if there is more than one way to do the integral. If there is more than one way we'll then need to determine which method we should use. The general rule of thumb that I use in my classes is that you should use the method that *you* find easiest. This may not be the method that others find easiest, but that doesn't make it the wrong method.

One of the more common mistakes with integration by parts is for people to get too locked into perceived patterns. For instance, all of the previous examples used the basic pattern of taking *u* to be the polynomial that sat in front of another function and then letting *dv* be the other function. This will not always happen so we need to be careful and not get locked into any patterns that we think we see.